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Aim. In this blurb we will explain Bhatt's observation that the cotangent complex satisfies descent along flat covers.[1]

Preliminaries. All rings are commutative. We let Δ be the usual simplex category, Ab the category of abelian groups and Sp the stable ∞-category of spectra. We will always work in cohomological grading i.e. differentials increase degree.

Let k be a ring and let Alg_k denote the category of k-algebras and let AffSch_k denote the opposite category of affine schemes over k. Let $D(k)$ be the derived ∞-category of k-modules. [2]

If A is a k-algebra, then the cotangent complex $L_{A/k}$ lives in $D(k)$. Moreover, the assignment $A \to L_{A/k}$ is functorial in A and so we can think of $L_{-/k}$ as a $D(k)$-valued presheaf on AffSch_k.

In what follows, we will explain Bhatt's proof of the fact that $L_{-/k}$ is in fact a sheaf on the latter category when we endow it with the flat topology.

Let us first recall what this means more generally. For any arrow $A \to B \in \mathrm{Alg}_k$ we can form the Cech co-nerve $\mathrm{Cech}(A \to B)_\bullet$, which is the cosimplicial commutative ring given by $\mathrm{Cech}(A \to B)_\bullet[n] = B^{\otimes_A^{n+1}}$ and obvious maps. From now on we will suppress the bullet and just write $\mathrm{Cech}(A \to B)$ for the co-nerve.

Let $F\colon \mathrm{Alg}_k \to \mathcal{C}$ be a functor into an ∞-category like $D(k)$ or Sp. For any cosimplicial object $C_\bullet \in \mathrm{Alg}_k$, we get a cosimplicial object $F(C_\bullet)$ in $\mathcal{C}$ obtained by applying F levelwise. In particular $F(\mathrm{Cech}(A \to B))$ is the cosimplicial object obtained by the above procedure (where we continue to suppress the bullet).

Definition 0.1. We will say that $F\colon \mathrm{Alg}_k \to \mathcal{C}$ satisfies descent relative to $A \to B$ if

$$F(A) \simeq \varprojlim_{\Delta} F(\mathrm{Cech}(A \to B)),$$

where the equivalence holds in $\mathcal{C}$.

If $\mathcal{C} = D(k)$ then we denote $\varprojlim_{\Delta}$ as Tot.

Flat descent for cotangent complexes. The cotangent complex is a functor $L_{-/k}\colon \mathrm{Alg}_k \to D(k)$ and from the above discussion we see that to show fpqc descent we need to show that for any faithfully flat morphism $A \to B$

$$L_{A/k} \simeq \mathrm{Tot}(L_{\mathrm{Cech}(A\to B)/k}).$$

Notice that the morphism $A \to B$ induces a morphism of cosimplicial k-algebras

$$k \to A \to \mathrm{Cech}(A \to B)$$

where we consider k and A as constant cosimplicial rings. Applying the cotangent complex we get the cosimplicial cofiber diagram in $D(\mathrm{Cech}(A \to B))$[3]

$$L_{A/k} \otimes_A^L \mathrm{Cech}(A \to B) \to L_{\mathrm{Cech}(A\to B)/k} \to L_{\mathrm{Cech}(A\to B)/A}. \tag{0.1}$$

Lemma 0.2. *The result follows from the following two conditions*

(1) The natural map $A \to \mathrm{Cech}(A \to B)$ *induces an equivalence*

$$L_{A/k} \simeq \mathrm{Tot}(L_{A/k} \otimes_A^L \mathrm{Cech}(A \to B)),$$

(2) $\mathrm{Tot}(L_{\mathrm{Cech}(A\to B)/A}) \simeq 0.$

[1]Thanks to Max Johnson for explaining almost all of the higher categorical machinery in this blurb. However, any mistakes regarding the higher machinery are solely the author's own.

[2]We note that here it is essential to work with higher categories. Without the ∞-enhancement, the notion of descent doesn't make sense since limits will not necessarily exist in the 1-categorical version of the derived category.

[3]This category should be interpreted as the analogue of the usual category with descent datum.

Proof. Taking Tot of the diagram in 0.1, we see that the second equivalence implies that

$$\mathrm{Tot}(L_{A/k} \otimes_A^L \mathrm{Cech}(A \to B)) \simeq \mathrm{Tot}\, L_{\mathrm{Cech}(A\to B)/k}$$

and the left term is identified with $L_{A/k}$ by the first equivalence, which then proves the result. $\square$

Before sketching the proof of the first condition, it is useful to review the Eilenberg-MacLane construction.

Construction 0.3 (Eilenberg-MacLane construction). There is a fully-faithful lax monoidal functor $\mathrm{Alg}_k \to \mathrm{Sp}$ which sends a k-algebra R to a discrete E_∞-ring which we also denote abusively as R. Here discreteness means that $\pi_i(R) = R$ when $i = 0$ and is 0 otherwise. In particular, such a ring is connective i.e. has homotopy only in non-negative degrees. Applying the functor to k itself allows us to view k as a E_∞-ring spectrum and thus view Alg_k as the full subcategory of E_∞-k-algebras spanned by discrete objects. Finally, using this embedding one identifies $D(R)$ with $\mathrm{Mod}(R)$ where the latter category is that of module spectra over the E_∞-k-algebra R. This identification is t-exact and preserves discrete objects.

Proposition 0.4. *There is a canonical equivalence*

$$L_{A/k} \simeq \mathrm{Tot}(L_{A/k} \otimes_A^L \mathrm{Cech}(A \to B)).$$

Proof. For any faithfully flat morphism $R \to S$ of connective E_∞-k-algebras (with k viewed as a spectrum), one can form the Cech co-nerve $\mathrm{Cech}(R \to S)$ as in the discrete case. By Lurie's augmentation of Grothendieck's descent theory, there is an equivalence $R \simeq \varprojlim_\Delta \mathrm{Cech}(R \to S)$ which remains an equivalence after tensoring with any module spectrum $M \in \mathrm{Mod}(R)$ i.e.

$$M \simeq \varprojlim_{\Delta} M \otimes_R \mathrm{Cech}(R \to S).$$

In our specific case, using the Construction 0.3 and identifying $L_{A/k} \in D(A)$ with it's image in $\mathrm{Mod}(A)$ we are done. $\square$

We will now explain the proof of the second condition in Lemma 0.2.

Proposition 0.5. *One has*

$$\mathrm{Tot}\, L_{\mathrm{Cech}(A\to B)/A} \simeq 0$$

Remark 0.6. Let us first review what we are trying to prove. Let $E_\bullet$ be a cosimplicial complex over A. Then $\mathrm{Tot}(E_\bullet) = \varprojlim_\Delta (E_\bullet)$ is the totalisation of the double complex associated to $E_\bullet$ by taking the product totalisation. There is a canonical spectral sequence whose E_2-page can be identified with the cohomology of the cochain complex associated to the cosimplicial abelian group $\pi_i(E_\bullet)$. More precisely we have $E_2^{i,j} = H^i(\pi_j(E_\bullet)) \Rightarrow \pi_{i+j}(\mathrm{Tot}(E))$. As a consequence if we can show that the cochain complexes $\pi_j(E_\bullet)$ are acyclic for each $j \in \mathbf{Z}$, then we are done.

Proof. By Remark 0.6 we will show that the cochain complexes associated to the cosimplicial abelian groups $\pi_j(L_{\mathrm{Cech}(A\to B)/A})$ are acyclic. Since $A \to B$ is faithfully flat, it is enough to show that $\pi_j(L_{\mathrm{Cech}(A\to B)/A} \otimes_A B)$ is acyclic. But by flat base change for cotangent complexes $L_{\mathrm{Cech}(A\to B)/A} \otimes_A B \simeq L_{(\mathrm{Cech}(A\to B)\otimes_A B)/B}$ and so we only have to show that $\pi_j(L_{(\mathrm{Cech}(A\to B)\otimes_A B)/B})$ is acyclic. Consider the morphism of cosimplicial rings $B \to \mathrm{Cech}(A \to B) \otimes_A B$ obtained by applying $- \otimes_A B$ to $A \to \mathrm{Cech}(A \to B)$. In degree 0 it corresponds to $B \to B \otimes_A B$ sending $b \to b \otimes 1$. This morphism has an evident section sending $b_1 \otimes b_2 \to b_1 b_2$ and hence by general nonsense corresponding to cosimplicial objects, is a homotopy equivalence. Since any functor preserves homotopy equivalences of cosimplicial objects, we see that $\pi_j(L_{\mathrm{Cech}(A\to B)\otimes_A B}/B)$ is homotopy equivalent to $\pi_j L_{B/B} \simeq 0$. Thus we win.

$\square$